

A class of symmetric Bell diagonal entanglement witnesses – a geometric perspective

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Abstract. We provide a class of Bell diagonal entanglement witnesses displaying an additional local symmetry – a maximal commutative subgroup of the unitary group $U(n)$. Remarkably, this class of witnesses is parameterized by a torus being a maximal commutative subgroup of an orthogonal group $SO(n-1)$. It is shown that a generic element from the class defines an indecomposable entanglement witness. The paper provides a geometric perspective for some aspects of the entanglement theory and an interesting interplay between group theory and block-positive operators in $\mathbb{C}^n \otimes \mathbb{C}^n$.

1. Introduction

Symmetry plays a prominent role in modern physics. In many cases it enables one to simplify the analysis of the corresponding problems and very often it leads to much deeper understanding and the most elegant mathematical formulation of the corresponding physical theory. In entanglement theory [1, 2] the idea of symmetry was first applied by Werner [3] to construct an important family of bipartite $n \otimes n$ quantum states which are invariant under the following local unitary operations

$$\rho \longrightarrow U \otimes U \rho (U \otimes U)^\dagger, \quad (1.1)$$

for any $U \in U(n)$, where $U(n)$ denotes the group of unitary $d \times d$ matrices. Another family of symmetric states (so called isotropic states) is governed by the following invariance rule

$$\rho \longrightarrow U \otimes U^* \rho (U \otimes U^*)^\dagger, \quad (1.2)$$

where U^* is the complex conjugate of U in some fixed orthonormal basis $\{e_0, \dots, e_{n-1}\}$ in \mathbb{C}^n (see [4, 5]). If we allow the full unitary group $U(n)$, then the only bipartite operators invariant under $U(n) \otimes U(n)$ are the identity operator $\mathbb{I}_n \otimes \mathbb{I}_n$ and the flip (or swap) operator \mathbb{F} defined by $\mathbb{F} x \otimes y = y \otimes x$. Similarly, the only bipartite operators invariant under $U(n) \otimes U(n)^*$ are the identity operator invariant $\mathbb{I}_n \otimes \mathbb{I}_n$ and the rank-1 projector

onto the maximally entangled state $P_n^+ = |\psi_n^+\rangle\langle\psi_n^+|$, where $|\psi_n^+\rangle = \frac{1}{\sqrt{n}} \sum_k e_k \otimes e_k$. One finds the following formulae for the Werner state

$$\rho_f = \frac{1}{n(n-f)} (\mathbb{I}_n \otimes \mathbb{I}_n - f\mathbb{F}) , \quad (1.3)$$

and isotropic state

$$\rho_p = \frac{1-p}{n^2} \mathbb{I}_n \otimes \mathbb{I}_n + pP_n^+ , \quad (1.4)$$

respectively. Remarkably, the properties of these two families of bipartite symmetric states are fully controlled by the operation of partial transposition: both ρ_f and ρ_p are separable iff they are PPT, i.e. $f \leq 1/n$ and $p \leq 1/(n+1)$ for Werner and isotropic state, respectively (a bipartite state ρ is PPT if its partial transposition ρ^Γ defines a positive operator). This example shows how symmetry simplifies separability problem in the entanglement theory. A general separability problem is much harder and the classification of states of a composite quantum system is very subtle [2, 6]. Let us recall that the most general approach to characterize quantum entanglement uses a notion of an entanglement witness. A Hermitian operator W acting in $\mathcal{H}_A \otimes \mathcal{H}_B$ is block-positive if $\langle x \otimes y | W | x \otimes y \rangle \geq 0$ for all $x \in \mathcal{H}_A$ and $y \in \mathcal{H}_B$. Clearly, a positive operator is necessarily block-positive but the converse needs not be true. An entanglement witness (a notion introduced by Terhal [7]) is a block-positive operator which is not positive, i.e. it possesses at least one negative eigenvalue (see a recent review [8] for detailed presentation). Remarkably, it turns out that any entangled state can be detected by some entanglement witness and hence the knowledge of witnesses enables us to perform full classification of states of composite quantum systems: a state ρ living in $\mathcal{H}_A \otimes \mathcal{H}_B$ is entangled iff there is an entanglement witness W such that $\text{tr}(\rho W) < 0$ [2]. An entanglement witness W is optimal [9] if there is no other witness which detects more entangled states than W . In the class of $U \otimes U$ -invariant EWs an optimal witness is provided by a flip operator

$$W = \mathbb{F} . \quad (1.5)$$

Similarly, an optimal $U \otimes U^*$ -invariant EW is provided by

$$W' = \mathbb{I}_n \otimes \mathbb{I}_n - nP_n^+ . \quad (1.6)$$

One easily finds that a Werner state ρ_f is entangled iff $\text{tr}(\mathbb{F}\rho_f) < 0$ and similarly an isotropic state ρ_p is entangled iff $\text{tr}(W'\rho_p) < 0$. Both witnesses (1.5) and (1.6) are decomposable, i.e. $W = A + B^\Gamma$, where $A, B \geq 0$ and B^Γ denotes a partial transposition of B . Decomposable EWs can not detect PPT entangled states. It should be stressed there is no universal method to construct an indecomposable EW which can be used to detect PPT entangled states.

It is, therefore, clear that to define a bigger class of symmetric states and entanglement witnesses one has to restrict the local symmetry from the full unitary group $U(n)$ to one of its subgroups. In this paper we consider $G \otimes G^*$ -invariant bipartite operators in $\mathbb{C}^n \otimes \mathbb{C}^n$, where G defines a subgroup of $U(n)$. Within a class of such $G \otimes G^*$ -invariant

operators we provide a detailed analysis of entanglement witnesses. Remarkably, a generic EW from this class is indecomposable and hence it may serve as a detector of bound entanglement. The paper provides a geometric perspective for some aspects of the entanglement theory and an interesting interplay between group theory and block-positive operators in $\mathbb{C}^n \otimes \mathbb{C}^n$.

2. A class of symmetric operators

Let us consider the following subgroup

$$G_1 = \{ U \in U(n) \mid U = \sum_{k=0}^{n-1} e^{i\phi_k} E_{kk} \} \subset U(n) , \quad (2.1)$$

where $E_{kl} := |e_k\rangle\langle e_l|$ and $\phi_k \in [0, 2\pi)$. Note, that G_1 is a maximal commutative subgroup of $U(n)$ (n -dimensional torus parameterized by angles ϕ_k). Now, the $(G_1 \otimes G_1^*)$ -invariant operator has the following form [10]

$$X = \sum_{k,l=0}^{n-1} a_{kl} E_{kk} \otimes E_{ll} + \sum_{k \neq l=0}^{n-1} b_{kl} E_{kl} \otimes E_{kl} . \quad (2.2)$$

Note, that X is Hermitian iff $a_{kl} \in \mathbb{R}$ and $b_{kl} = b_{lk}^*$. Consider now a discrete subgroup

$$G_2 = \{ \lambda^m U_{kl} \mid k, l, m = 0, 1, \dots, n-1 \} \subset U(n) , \quad (2.3)$$

where $\lambda = e^{2\pi i/d}$ and U_{kl} denotes a family of unitary Weyl operators defined as follows [11, 12, 13]

$$U_{mk} e_l = \lambda^{ml} e_{l+k} , \quad \text{mod } n . \quad (2.4)$$

The matrices U_{kl} satisfy

$$U_{kl} U_{rs} = \lambda^{ks} U_{k+r, l+s} , \quad U_{kl}^* = U_{-k, l} , \quad U_{kl}^\dagger = \lambda^{kl} U_{-k, -l} , \quad (2.5)$$

and the following orthogonality relations

$$\text{tr}(U_{kl} U_{rs}^\dagger) = n \delta_{kr} \delta_{ls} . \quad (2.6)$$

One has therefore

$$G_2 \otimes G_2^* = \{ U_{kl} \otimes U_{-k, l} \mid k, l = 0, 1, \dots, n-1 \} . \quad (2.7)$$

Note, that $G_2 \otimes G_2^*$ defines a discrete commutative subgroup of $U(n) \otimes U^*(n)$. Interestingly, its commutant, that is, an algebra of $G_2 \otimes G_2^*$ -invariant operators is spanned by $U_{kl} \otimes U_{-k, l}$ and hence any $G_2 \otimes G_2^*$ -invariant operator has the following form

$$X = \sum_{k,l=0}^{n-1} c_{kl} U_{kl} \otimes U_{-k, l} . \quad (2.8)$$

Note, that (2.8) defines a Hermitian operator iff

$$c_{kl} = c_{n-k, n-l}^* . \quad (2.9)$$

Denote by $|\psi_{kl}\rangle$ generalized Bell states in $\mathbb{C}^n \otimes \mathbb{C}^n$

$$|\psi_{kl}\rangle = \mathbb{I}_n \otimes U_{kl} |\psi_n^+\rangle , \quad (2.10)$$

and let $P_{kl} = |\psi_{kl}\rangle\langle\psi_{kl}|$ be the corresponding rank-1 projectors. One easily shows that P_{kl} span the entire commutant of $G_2 \otimes G_2^*$ and hence any $G_2 \otimes G_2^*$ -invariant operator is Bell diagonal, that is, it can be represented as follows

$$X = \sum_{k,l=0}^{n-1} x_{kl} P_{kl} . \quad (2.11)$$

One easily finds

Lemma 2.1 *A Hermitian $G_1 \otimes G_1^*$ -invariant operator (2.2) is $G_2 \otimes G_2^*$ -invariant if the matrix a_{kl} is circulant, that is, $a_{kl} = \alpha_{k-l} \in \mathbb{R}$, and $b_{kl} = c \in \mathbb{R}$.*

Similarly,

Lemma 2.2 *A Hermitian $G_2 \otimes G_2^*$ -invariant operator (2.8) is $G_1 \otimes G_1^*$ -invariant if the matrix c_{kl} has the following structure*

$$c_{kl} = \begin{pmatrix} c_0 & c & \dots & c \\ c_1 & c & \dots & c \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c & \dots & c \end{pmatrix} , \quad (2.12)$$

where ‘ c ’ is an arbitrary real parameter and the vector $c_k := c_{k0}$ is defined as follows

$$c_k = \sum_{l=0}^{n-1} \omega^{-kl} \alpha_l , \quad (2.13)$$

that is, it is a discrete Fourier transform of a real vector α_l .

It is, therefore, clear that two representations (2.2) and (2.8) are complementary to each other. Now, combining (2.2) and (2.8) we obtain the following formula for a spectral resolution of any $G_1 \otimes G_1^*$ -invariant Bell diagonal operator

$$X = (\alpha_0 + 1)\Pi_0 + \sum_{k=1}^{n-1} \alpha_k \Pi_k + \beta n P_n^+ , \quad (2.14)$$

where

$$\Pi_k = P_{0k} + P_{1k} + \dots + P_{n-1,k} , \quad k = 0, \dots, n-1 . \quad (2.15)$$

Now, if (2.14) represents an EW then necessarily $\alpha_k \geq 0$ for $k = 0, \dots, n-1$ and $\beta < 0$. From now on we fix $\beta = -1$. Clearly, these conditions are necessary but not sufficient. We pose the following question: what are the additional properties of $\{\alpha_0, \dots, \alpha_{n-1}\}$ which guarantee that the formula (2.14) provides a legitimate entanglement witness. Note, that if $\alpha_0 = 0$ and $\alpha_1 = \dots = \alpha_{n-1} = 1$, then (2.14) reconstructs (1.6). The class of witnesses

$$W[\alpha_0, \dots, \alpha_{n-1}] := (\alpha_0 + 1)\Pi_0 + \sum_{k=1}^{n-1} \alpha_k \Pi_k - n P_n^+ , \quad (2.16)$$

seems to be very special, however, it turns out that many EWs considered in the literature belong to this class.

3. Entanglement witnesses vs. positive maps

Due to the Choi-Jamiołkowski isomorphism any entanglement witness W in $\mathbb{C}^n \otimes \mathbb{C}^n$ corresponds to a positive map $\Lambda : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ via the following relation

$$W = \sum_{i,j=0}^{n-1} E_{ij} \otimes \Lambda(E_{ij}) . \quad (3.1)$$

The map corresponding to (2.16) has the following form

$$\begin{aligned} \Lambda(E_{ii}) &= \sum_{j=0}^{n-1} a_{ij} E_{jj} , \\ \Lambda(E_{ij}) &= -E_{ij} , \quad i \neq j , \end{aligned} \quad (3.2)$$

where $a_{ij} := \alpha_{i-j} \geq 0$.

Proposition 3.1 *A linear map Λ is positive if and only if the following cyclic inequalities*

$$\sum_{i=0}^{n-1} \frac{t_i^2}{(\alpha_0 + 1)t_i^2 + \sum_{k=1}^{n-1} \alpha_k t_{i+k}^2} \leq 1 . \quad (3.3)$$

are satisfied for all $t_0, t_1, \dots, t_n \geq 0$. Λ is completely positive if and only if $\alpha_0 \geq n - 1$.

In particular taking $t_0 = \dots = t_{n-1}$ one finds

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1} \geq n - 1 . \quad (3.4)$$

Hence, if $W[\alpha_0, \dots, \alpha_{n-1}]$ is an entanglement witness, then necessarily $\{\alpha_0, \dots, \alpha_{n-1}\}$ satisfy (3.4) and additionally

$$0 \leq \alpha_0 < n - 1 . \quad (3.5)$$

Interestingly, one has

Proposition 3.2 *For $n = 2$ conditions (3.4) and (3.5) are necessary and sufficient.*

However, for $n \geq 3$ these conditions are no longer sufficient. For $n = 3$ introducing $a = \alpha_0$, $b = \alpha_1$ and $c = \alpha_2$ one has the following well known result

Theorem 3.1 ([16]) *An operator $W[a, b, c]$ is an entanglement witness if and only if apart from (3.4) and (3.5) the following extra condition has to be satisfied: if $a \leq 1$, then*

$$bc \geq (1 - a)^2 . \quad (3.6)$$

Moreover, being an entanglement witness it is indecomposable if and only if

$$4bc < (2 - a)^2 . \quad (3.7)$$

From now on we consider entanglement witnesses $W[\alpha_0, \dots, \alpha_{n-1}]$ which belong to the boundary of a set of entanglement witnesses. Clearly, any optimal witness belongs to this boundary. Note, that the corresponding parameters $\{\alpha_0, \dots, \alpha_{n-1}\}$ instead of (3.4) satisfy the following equality

$$\alpha_0 + \alpha_1 + \dots + \alpha_{n-1} = n - 1 . \quad (3.8)$$

Now, for $n = 3$ we look for a set of parameters $a, b, c \geq 0$ belonging to a simplex $a + b + c = 2$ and satisfying for $a \leq 1$

$$bc = (1 - a)^2 , \quad (3.9)$$

which corresponds to the boundary of a set defined by an inequality (3.6). Actually, the above condition defines an ellipse $bc = (b + c - 1)^2$ on the bc -plane (cf. [14]). It is easy to show that the above conditions, i.e.

$$a + b + c = 2 , \quad bc = (1 - a)^2 , \quad (3.10)$$

are equivalent to much more symmetric ones

$$a + b + c = 2 , \quad a^2 + b^2 + c^2 = 2 . \quad (3.11)$$

Now, the intersection of the 2D sphere $a^2 + b^2 + c^2 = 2$ and the plane $a + b + c = 2$ defines a circle and its projection on the bc -plane gives rise to an ellipse $bc = (b + c - 1)^2$ (cf. Fig. 1). Note, that equivalently one may describe the above circle as an intersection of the following sphere centered at $(1, 1, 1)$

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 = 1 , \quad (3.12)$$

or the one centered at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ (the middle of the simplex)

$$\left(a - \frac{2}{3}\right)^2 + \left(b - \frac{2}{3}\right)^2 + \left(c - \frac{2}{3}\right)^2 = \frac{2}{3} , \quad (3.13)$$

with a plane $a + b + c = 2$.

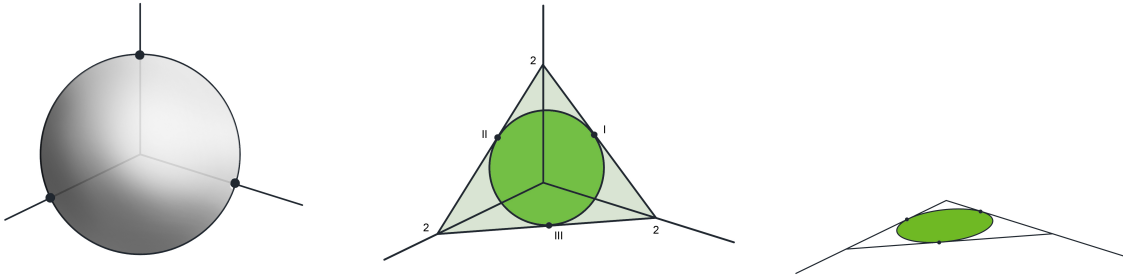


Figure 1. On the left: a 2D sphere $a^2 + b^2 + c^2 = 2$. On the middle: the intersection of a sphere and a simplex $a + b + c = 2$. On the right: an ellipse on a bc -plane being a projection of a circle. Characteristic points: I and II correspond to Choi maps and III to the reduction map.

It should be stressed that for $n > 3$ we do not know the complete set of conditions implied by the cyclic inequalities (3.3) (see [15] for partial results for $n = 4$).

4. Witnesses parameterized by an orthogonal group

In this section we analyze a class of witnesses $W[\alpha_0, \dots, \alpha_{n-1}]$ generated by a certain family of positive maps proposed in [17]: let us define a set of Hermitian traceless matrices

$$F_\ell = \frac{1}{\sqrt{\ell(\ell+1)}} \left(\sum_{k=0}^{\ell-1} E_{kk} - \ell E_{\ell\ell} \right), \quad \ell = 1, \dots, n-1. \quad (4.1)$$

One defines a real $n \times n$ matrix

$$a_{ij} = \frac{n-1}{n} + \sum_{\alpha, \beta=1}^{n-1} \langle e_i | F_\alpha | e_i \rangle R_{\alpha\beta} \langle e_j | F_\beta | e_j \rangle, \quad (4.2)$$

where $R_{\alpha\beta}$ is an orthogonal $(n-1) \times (n-1)$ orthogonal matrix. Due to the fact that F_α is traceless for $\alpha = 1, \dots, n-1$, one finds

$$\sum_{i=1}^{n-1} a_{ij} = \sum_{j=1}^{n-1} a_{ij} = n-1, \quad (4.3)$$

Moreover, it turns out [17] that matrix elements $a_{ij} \geq 0$ and hence

$$\tilde{a}_{ij} := \frac{1}{n-1} a_{ij}, \quad (4.4)$$

defines a doubly stochastic matrix. Consider now a linear map Λ defined by (3.2) with a_{ij} defined by (4.2).

Proposition 4.1 ([17]) *For any orthogonal matrix $R_{\alpha\beta}$ a linear map Λ is positive.*

Suppose we are given a $n \times n$ matrix a_{ij} such that $a_{ij} \geq 0$ and (4.3) is satisfied.

Proposition 4.2 ([18]) *A matrix a_{ij} can be represented by (4.2) if and only if*

$$\sum_{k=0}^{n-1} a_{ik} a_{jk} = \delta_{ij} + n-2, \quad (4.5)$$

for $i, j = 0, \dots, n-1$.

Hence, if the matrix a_{ij} is circulant, i.e. $a_{ij} = \alpha_{i-j}$, then (4.5) implies the following set of conditions for a set of parameters $\{\alpha_0, \dots, \alpha_{n-1}\}$:

$$\sum_{k=0}^{n-1} \alpha_{i-k} \alpha_{j-k} = \delta_{ij} + n-2. \quad (4.6)$$

Example 4.1 *For $n = 3$ using again the following notation $a = \alpha_0$, $b = \alpha_1$ and $c = \alpha_2$ the formula (4.6) implies*

$$a^2 + b^2 + c^2 = 2, \quad \text{for } i = j, \quad (4.7)$$

and

$$ac + ba + cb = 1, \quad \text{for } i \neq j. \quad (4.8)$$

Note, however, that (4.11) and (4.8) are not independent. Indeed, taking into account $a + b + c = 2$ one has

$$4 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ac + ba + cb) ,$$

and hence (4.11) implies (4.8). One concludes, therefore, that this class is fully characterized by

$$a + b + c = 2 , \quad a^2 + b^2 + c^2 = 2 , \quad (4.9)$$

which reproduce (3.11).

Example 4.2 For $n = 4$ using $a = \alpha_0$, $b = \alpha_1$, $c = \alpha_2$ and $d = \alpha_3$ one has

$$a + b + c + d = 3 , \quad (4.10)$$

and the formula (4.6) implies

$$a^2 + b^2 + c^2 + d^2 = 3 , \quad ac + bd = 1 , \quad (a + c)(b + d) = 2 . \quad (4.11)$$

Actually, assuming (4.10) only two of the above three conditions are independent. Introducing $x = a + c$ and $y = b + d$ one obtains the following equations for a pair (x, y) :

$$xy = 2 , \quad x^2 + y^2 = 5 ,$$

with two solutions $(x = 1, y = 2)$ and $(x = 2, y = 1)$. Finally, we have two classes of admissible parameters $\{a, b, c, d\}$ constrained by

$$a + b + c + d = 3 , \quad a^2 + b^2 + c^2 + d^2 = 3 , \quad b + d = 1 , \quad (4.12)$$

and

$$a + b + c + d = 3 , \quad a^2 + b^2 + c^2 + d^2 = 3 , \quad b + d = 2 . \quad (4.13)$$

Note, that the intersection of a 3D sphere $a^2 + b^2 + c^2 + d^2 = 3$ with a simplex $a + b + c + d = 3$ may be equivalently rewritten as the intersection with the following sphere centered at $(1, 1, 1, 1)$

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 + (d - 1)^2 = 1 , \quad (4.14)$$

or the one centered at the middle of the simplex $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})$

$$\left(a - \frac{3}{4}\right)^2 + \left(b - \frac{3}{4}\right)^2 + \left(c - \frac{3}{4}\right)^2 + \left(d - \frac{3}{4}\right)^2 = \frac{3}{4} , \quad (4.15)$$

which provide analogs of (3.12) and (3.13), respectively.

Clearly, for higher dimensions the number of conditions implied by (4.6) grows: one always has

$$\sum_{k=0}^{n-1} \alpha_k^2 = n - 1 , \quad (4.16)$$

corresponding to $i = j$ plus some extra conditions following from (4.6) for $i \neq j$. The above conditions define $(n - 2)$ -dim. sphere as an intersection of $(n - 1)$ -dim. sphere (4.18) with the simplex $\sum_{k=0}^{n-1} \alpha_k = n - 1$. The same intersection is provided by

$$\sum_{k=0}^{n-1} [\alpha_k - 1]^2 = 1 , \quad (4.17)$$

and

$$\sum_{k=0}^{n-1} \left(\alpha_k^2 - \frac{n-1}{n} \right)^2 = \frac{n-1}{n} , \quad (4.18)$$

in analogy with (4.14) and (4.15), respectively.

5. Witnesses constructed from Weyl operators

Now, we provide characterization of entanglement witnesses from the previous section using a complementary representation (2.8). Authors of [13] provided the following

Proposition 5.1 *Let W be a Hermitian Bell diagonal operator defined by*

$$W = a \sum_{k,l=0}^{n-1} c_{kl} U_{kl} \otimes U_{-k,l} , \quad (5.1)$$

with $a > 0$, $c_{00} = n - 1$. If $|c_{kl}| \leq 1$ (apart from c_{00}), then W is block positive.

Using Lemma 2.2 one easily finds that formula (5.1) reproduces $W = W[\alpha_0, \dots, \alpha_{n-1}]$ iff

$$a = \frac{1}{n} , \quad c_{kl} = 1 , \quad l = 1, \dots, n-1 , \quad (5.2)$$

and

$$c_{k0} = \sum_{l=0}^{n-1} \omega^{-kl} \alpha_l . \quad (5.3)$$

Note that formula (5.3) implies $c_{00} = \alpha_0 + \dots + \alpha_{n-1} = n - 1$. Interestingly, a set of conditions (4.6) for parameters α_k is equivalent to a set of remarkably simple conditions for parameters c_{k0} .

Proposition 5.2 *A set $\{\alpha_0, \dots, \alpha_{n-1}\}$ such that $\alpha_k \geq 0$ and $\alpha_0 + \dots + \alpha_{n-1} = n - 1$ satisfies (4.6) if and only if a set of c_{k0} defined by (5.3) satisfies*

$$c_{00} = n - 1 , \quad c_{k0} = c_{n-k,0}^* , \quad |c_{k0}| = 1 , \quad (5.4)$$

for $k = 1, \dots, n - 1$.

Proof: one has

$$c_{n-k,0} = \sum_{l=0}^{n-1} \omega^{-(n-k)l} \alpha_l = \sum_{l=0}^{n-1} \omega^{-nl} \omega^{kl} \alpha_l = c_{k0}^* , \quad (5.5)$$

due to $\omega^{nl} = 1$. Now, the inverse to (5.3) reads

$$\alpha_k = \frac{1}{n} \sum_{l=0}^{n-1} \omega^{kl} c_{l0} . \quad (5.6)$$

Suppose now that $|c_{k0}| = 1$. Using the fact that $\alpha_k = \alpha_k^*$ one has

$$\begin{aligned} \sum_{k=0}^{n-1} \alpha_k \alpha_k^* &= \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \omega^{ik} c_{i0} \sum_{j=0}^{n-1} \omega^{-jk} c_{j0} = \frac{1}{n^2} \sum_{i,j=0}^{n-1} \left(\sum_{k=0}^{n-1} \omega^{(i-j)k} \right) c_{i0} c_{j0}^* \\ &= \frac{1}{n} \sum_{i=0}^{n-1} |c_{i0}|^2 = \frac{1}{n} [(n-1)^2 + (n-1)] = n-1 , \end{aligned} \quad (5.7)$$

where we have used

$$\sum_{k=0}^{n-1} \omega^{(i-j)k} = n \delta_{ij} . \quad (5.8)$$

Similarly, for $i \neq j$

$$\begin{aligned} \sum_{k=0}^{n-1} \alpha_{i-k} \alpha_{j-k}^* &= \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{r=0}^{n-1} \omega^{r(i-k)} c_{r0} \sum_{s=0}^{n-1} \omega^{-s(j-k)} c_{s0}^* \\ &= \frac{1}{n^2} \left(\sum_{k=0}^{n-1} \omega^{(s-r)k} \right) \sum_{r,s=0}^{n-1} \omega^{ri} c_{r0} \omega^{-sj} c_{s0}^* = \frac{1}{n} \sum_{r=0}^{n-1} \omega^{r(i-j)} |c_{r0}|^2 \\ &= \frac{1}{n} ([n-1]^2 - 1) = n-2 , \end{aligned} \quad (5.9)$$

which proves (4.6). Conversely, if (4.6) is satisfied, then in a similar way one shows that $|c_{k0}|^2 = 1$. \square

Hence the entire class of witnesses is parameterized by phases of $c_{k0} = e^{i\phi_k}$. Due to $c_{k0} = c_{n-k,0}^*$ one has two cases:

- (i) if $n = 2m + 1$, then we have m independent phases $c_{10} = e^{i\phi_1}, \dots, c_{m0} = e^{i\phi_m}$.
- (ii) if $n = 2m + 2$, then we have m independent phases $c_{10} = e^{i\phi_1}, \dots, c_{m0} = e^{i\phi_m}$ and one real parameter $c_{m+1,0} = \pm 1$.

It shows that for an odd n ($n = 2m + 1$) the space of witnesses is parameterized by m -dim. torus \mathbb{T}_m and if n is even ($n = 2m + 2$) we have two classes of witnesses: each one corresponding to \mathbb{T}_m .

Remark 5.1 A similar observation holds for PPT Bell diagonal states, i.e. the structure of PPT Bell diagonal states in $\mathbb{C}^n \otimes \mathbb{C}^n$ depends upon the parity of 'n' (cf. [11]).

Example 5.1 For $n = 3$ putting $c_{10} = e^{i\phi} = c_{20}^*$ one finds

$$\begin{aligned} a &= \frac{1}{3}(2 + c_{10} + c_{10}^*) = \frac{2}{3}(1 + \cos \phi) , \\ b &= \frac{1}{3}(2 + \omega c_{10} + \omega^* c_{10}^*) = \frac{1}{3}(2 - \cos \phi - \sqrt{3} \sin \phi) , \\ c &= \frac{1}{3}(2 + \omega^* c_{10} + \omega c_{10}^*) = \frac{1}{3}(2 - \cos \phi + \sqrt{3} \sin \phi) , \end{aligned} \quad (5.10)$$

due to $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + i\sqrt{3})$.

Example 5.2 For $n = 4$ if $c_{10} = e^{i\phi} = c_{30}^*$ and $c_{20} = 1$ one finds

$$a = \frac{1}{2}(2 + \cos \phi), \quad b = \frac{1}{2}(1 - \sin \phi), \quad c = \frac{1}{2}(2 - \cos \phi), \quad d = \frac{1}{2}(1 + \sin \phi), \quad (5.11)$$

and similarly if $c_{10} = e^{i\psi} = c_{30}^*$ and $c_{20} = -1$ one has

$$a = \frac{1}{2}(1 + \cos \psi), \quad b = \frac{1}{2}(2 - \sin \psi), \quad c = \frac{1}{2}(1 - \cos \psi), \quad d = \frac{1}{2}(2 + \sin \psi). \quad (5.12)$$

Note, that for $c_{20} = 1$ one has $b + d = 1$, whereas for $c_{20} = -1$ one has $b + d = 2$. This way we reproduced two classes from Example 4.2.

It should be clear that the structure of tori \mathbb{T}_m is related with the properties of orthogonal group considered in the previous section. Note, that the structure of the orthogonal group differs in certain aspects between even and odd dimensions. For example, the reflection corresponding to ‘ $-\mathbb{I}$ ’ is orientation-preserving in even dimensions, but orientation-reversing in odd dimensions.

- (i) If $n = 2m + 1$, then $O(n - 1) = O(2m)$ and a single torus \mathbb{T}_m corresponds to a maximal commutative subgroup of $SO(2m)$.
- (ii) If $n = 2m + 2$, then $O(n - 1) = O(2m + 1)$ and we have two tori \mathbb{T}_m and \mathbb{T}'_m . Torus \mathbb{T}_m corresponds to a maximal commutative subgroup of $SO(2m + 1)$ whereas \mathbb{T}'_m is defined by composing \mathbb{T}_m with reflection, that is, $g \in \mathbb{T}'_m$ iff $-g \in \mathbb{T}_m$.

Remark 5.2 It should be stressed that a set $\{\alpha_0, \dots, \alpha_{n-1}\}$ satisfying (4.6) provides only a proper subset of admissible parameters. Note that

$$\alpha_0 \leq 2 \frac{n-1}{n} < 2, \quad (5.13)$$

and hence one can not reproduce well known entanglement witnesses corresponding to

$$\alpha_0 = n - k, \quad \alpha_1 = \dots = \alpha_{k-1} = 1, \quad \alpha_k = \dots = \alpha_{n-1} = 0, \quad (5.14)$$

for $k = 2, \dots, n - 2$.

6. Decomposability and optimality

Finally, we address the problem of decomposability of $W[\alpha_0, \dots, \alpha_{n-1}]$.

Theorem 6.1 An entanglement witness is decomposable if and only if $\alpha_k = \alpha_{n-k}$ for $k = 1, \dots, n - 1$.

Corollary 6.1 A generic $W[\alpha_0, \dots, \alpha_{n-1}]$ provides an indecomposable EW.

Proof of the Theorem: suppose that for some $k \in \{1, \dots, n - 1\}$ one has $\alpha_k > 0$ and $\alpha_k \neq \alpha_{n-k}$. We construct a PPT state ρ_ϵ such that $\text{tr}(\rho_\epsilon W[\alpha_0, \dots, \alpha_{n-1}]) < 0$ and hence we show that $W[\alpha_0, \dots, \alpha_{n-1}]$ is indecomposable. Let us consider the following operator

$$\rho_\epsilon = \left[\sum_{l=1}^{n-1} \Pi_l - \Pi_k - \Pi_{n-k} \right] + \epsilon \Pi_k + \frac{1}{\epsilon} \Pi_{n-k} + n P_n^+, \quad (6.1)$$

with $\epsilon > 0$. One easily check that both ρ and ρ^Γ are positive and hence ρ represents an unnormalized PPT state. One has

$$\begin{aligned} \text{tr}(\rho_\epsilon W[\alpha_0, \dots, \alpha_{n-1}]) &= n\epsilon\alpha_k + n\frac{1}{\epsilon}\alpha_{n-k} + n\left(\sum_{j=0}^{n-1} \alpha_j - \alpha_k - \alpha_{n-k}\right) - n(n-1) \\ &= n\left(\epsilon\alpha_k + \frac{1}{\epsilon}\alpha_{n-k} - (\alpha_k + \alpha_{n-k})\right). \end{aligned} \quad (6.2)$$

Hence $\text{tr}(\rho_\epsilon W[\alpha_0, \dots, \alpha_{n-1}]) < 0$ if $\epsilon \in (\epsilon_-, \epsilon_+)$, where

$$\epsilon_\pm = \frac{\alpha_k + \alpha_{n-k} \pm |\alpha_k - \alpha_{n-k}|}{\alpha_k}. \quad (6.3)$$

It is, therefore, clear that if $\alpha_k \neq \alpha_{n-k}$, then $\epsilon_+ > \epsilon_-$ and one can always find a suitable ϵ such that $\text{tr}(\rho_\epsilon W[\alpha_0, \dots, \alpha_{n-1}]) < 0$. To prove the converse let us assume that $\alpha_k = \alpha_{n-k}$. Note that

$$W[\alpha_0, \dots, \alpha_{n-1}] = P[\alpha_0, \dots, \alpha_{n-1}] + Q[\alpha_0, \dots, \alpha_{n-1}]^\Gamma, \quad (6.4)$$

where

$$P[\alpha_0, \dots, \alpha_{n-1}] = \sum_{k=1}^{n-1} \alpha_k |e_k \otimes e_{n-k} + e_{n-k} \otimes e_k\rangle \langle e_k \otimes e_{n-k} + e_{n-k} \otimes e_k|,$$

and

$$Q[\alpha_0, \dots, \alpha_{n-1}] = \sum_{i,j=1}^{n-1} Q_{ij} E_{ij} \otimes E_{ij},$$

where Q_{ij} is a circulant matrix such that $Q_{00} = \alpha_0$ and $Q_{0k} = \alpha_k - 1$ for $k > 0$. To show that $W[\alpha_0, \dots, \alpha_{n-1}]$ is indecomposable one has to prove that $P[\alpha_0, \dots, \alpha_{n-1}]$ and $Q[\alpha_0, \dots, \alpha_{n-1}]$ are positive matrices. Positivity of $P[\alpha_0, \dots, \alpha_{n-1}]$ is guaranteed by $\alpha_k \geq 0$. Now, the positivity of $Q[\alpha_0, \dots, \alpha_{n-1}]$ is equivalent to positivity of a circulant matrix Q_{ij} . The eigenvalues of Q_{ij} read

$$\lambda_j = \alpha_0 + \sum_{k=1}^{n-1} (\alpha_k - 1) \omega^{-jk}, \quad (6.5)$$

for $j = 0, \dots, n-1$. One finds $\lambda_0 = 0$ and for $j > 0$

$$\lambda_j = \alpha_0 + 1 + \sum_{k=1}^{n-1} \alpha_k \omega^{-jk} = c_{j0} + 1, \quad (6.6)$$

where we used (2.13). Note, that condition $\alpha_k = \alpha_{n-k}$ guarantees that all $c_j \in \mathbb{R}$ and hence since $|c_{j0}| = 1$ one has $c_{j0} = \pm 1$ and hence $\lambda_0 = n-1$ and $\lambda_j \in \{0, 2\}$ for $j > 0$ which proves positivity of Q_{ij} . \square

Corollary 6.2 $W[\alpha_0, \dots, \alpha_{n-1}]$ is decomposable if and only if $c_{k0} = \pm 1$ for $k = 1, \dots, n-1$.

Example 6.1 Taking $c_{k0} = -1$ one finds $\alpha_0 = 0$ and $\alpha_k = 1$ for $k > 0$. This way one reproduces an entanglement witness corresponding to the reduction map.

If $\Lambda : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$ is a linear map, then the dual map $\Lambda^\# : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is defined by

$$\text{tr}[\Lambda^\#(B)] := \text{tr}[\Lambda(A)] , \quad (6.7)$$

for any $A, B \in M_n(\mathbb{C})$. If W is a bipartite operators corresponding to Λ via (3.1), then denote by $W^\#$ an operator corresponding to $\Lambda^\#$. One finds

$$W[\alpha_0, \alpha_1 \dots, \alpha_{n-1}]^\# = W[\alpha_0, \alpha_{n-1}, \dots, \alpha_1] . \quad (6.8)$$

Interestingly, one has the following relation

$$W[\alpha_0, \alpha_1 \dots, \alpha_{n-1}]^\# = \mathbb{F} W[\alpha_0, \alpha_1, \dots, \alpha_{n-1}] \mathbb{F} , \quad (6.9)$$

where \mathbb{F} denotes a flip operator.

Corollary 6.3 *An entanglement witness $W[\alpha_0, \alpha_1 \dots, \alpha_{n-1}]$ is decomposable if and only if $W[\alpha_0, \alpha_1 \dots, \alpha_{n-1}]^\# = W[\alpha_0, \alpha_1 \dots, \alpha_{n-1}]$ or, equivalently, if the corresponding positive map Λ is self-dual, i.e. $\Lambda^\# = \Lambda$.*

Interestingly, for $n = 3$ it was shown [19, 20] that if $a \leq 1$, then $W[a, b, c]$ provides a set of optimal witnesses. Optimality of $W[\alpha_0, \dots, \alpha_{n-1}]$ for $n > 3$ deserves further studies.

7. Conclusions

We analyzed a class of Bell diagonal entanglement witnesses displaying an additional $G_1 \otimes G_1^*$ -symmetry. This class is characterized by a set of parameters $\{\alpha_0, \dots, \alpha_{n-1}\}$ satisfying a family of conditions. Interestingly, when transformed via discrete Fourier transform it gives rise to a family of complex coefficients c_{k0} satisfying remarkably simple conditions, that is, $|c_{k0}| = 1$ for $k = 1, \dots, n-1$. It proves that the family of entanglement witnesses is characterized by a torus $\{\phi_1, \dots, \phi_m\}$, where $c_{k0} = e^{i\phi_k}$ and $m = \lfloor n/2 \rfloor$. Actually, if n is odd there is only one torus, however, if n is even there are two tori. Interestingly, the structure of these tori corresponds to properties of orthogonal groups – torus provides a maximal abelian subgroup of $SO(n-1)$. Finally, we showed that a generic element from the class defines an indecomposable entanglement witness. Optimality of $W[\alpha_0, \dots, \alpha_{n-1}]$ for $n > 3$ provides an interesting open problem.

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